Motivations & Objectives

- Coastal forests – non-intrusive (natural) protection against ocean waves
- How effective can coastal vegetation dissipate incoming wave energy?
- Interactions between waves and vegetation:
  - Physical modeling – rigid/flexible cylinders or live vegetation (Wu et al. 2011, Maza et al. 2015)
  - Numerical modeling – N-S models, depth-integrated models (NLSW, Boussinesq-type equations)
  - Mathematical modeling – Homogenization theory (Mei et al. 2011, 2014)
- Previous work:
  - Develop a model to estimate wave attenuation by coastal forests of arbitrary shape
  - Linear model vs. experimental data (Liu et al. 2015, Chang et al. 2017a, b)



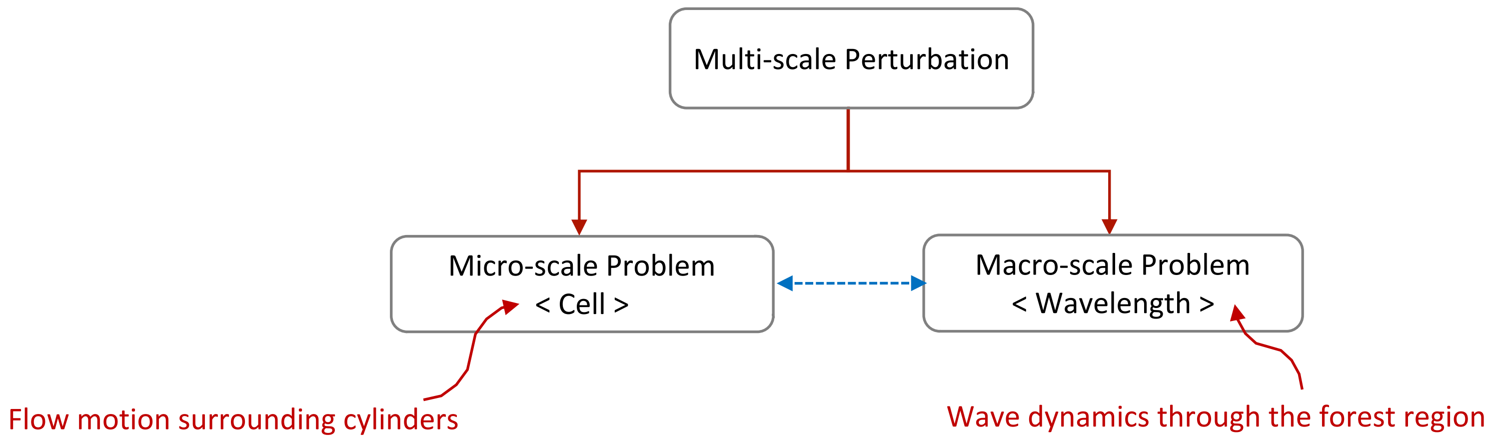
How about nonlinearity?

Motivations & Objectives

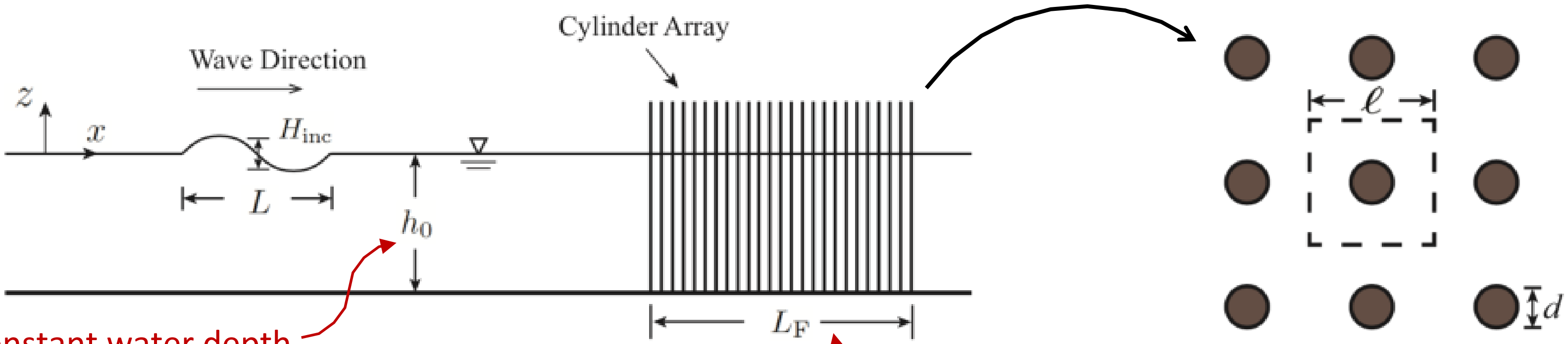
- Previous work:
  - Develop a model to estimate wave attenuation by coastal forests of arbitrary shape
  - Linear model vs. experimental data (Liu et al. 2015, Chang et al. 2017a, b)

How about nonlinearity?

- Extend the linear model (Mei et al. 2011) – homogenization theory
- Consider the effects of weak nonlinearity
- Investigate the nonlinear effects and harmonic generation



Periodic shallow-water waves



Constant water depth

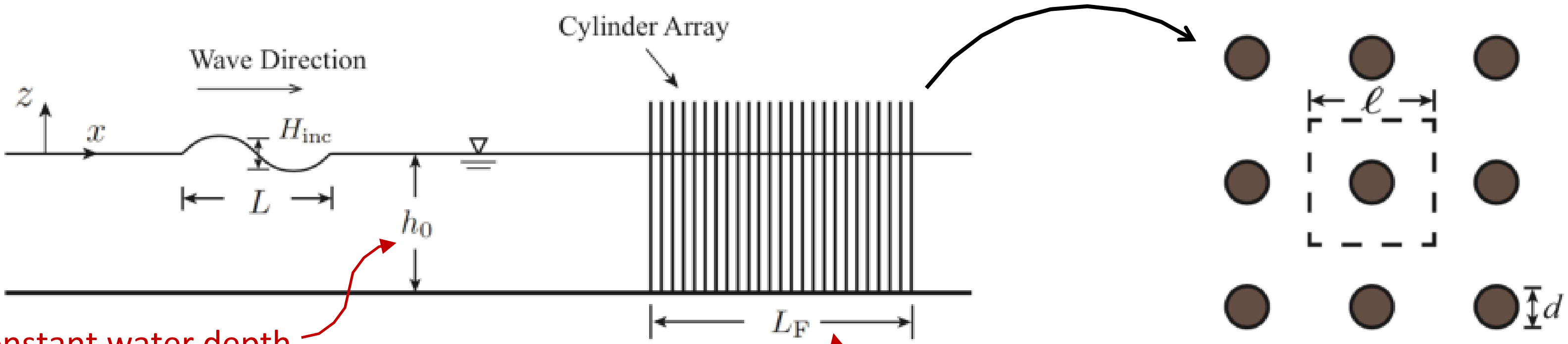
Emergent, rigid and vertical cylinders

Regular arrangements

- Conditions:
  - ➢ Shallow water: wavelength is much greater than water depth
  - ➢ Tree spacing is much smaller than the wavelength
  - ➢ Incident waves: simple-harmonic waves with weak nonlinearity

$$\ell \ll h_0 \ll 1/k_{inc}$$

Periodic shallow-water waves



Constant water depth

Emergent, rigid and vertical cylinders

Regular arrangements

- Governing equations (shallow water):

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_i}\left[u_i\left(h_0+\eta\right)\right] = 0, \quad i = 1,\ 2$$

$$\frac{\partial u_i}{\partial t} + \boxed{u_j \frac{\partial u_i}{\partial x_j}} = -g\frac{\partial \eta}{\partial x_i} + \nu_e \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad i \,\&\, j = 1,\ 2$$

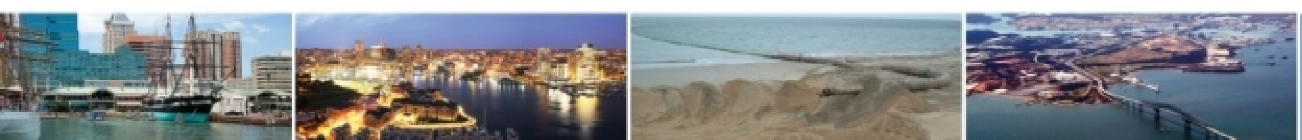
u_i : velocity components

η : free surface elevation

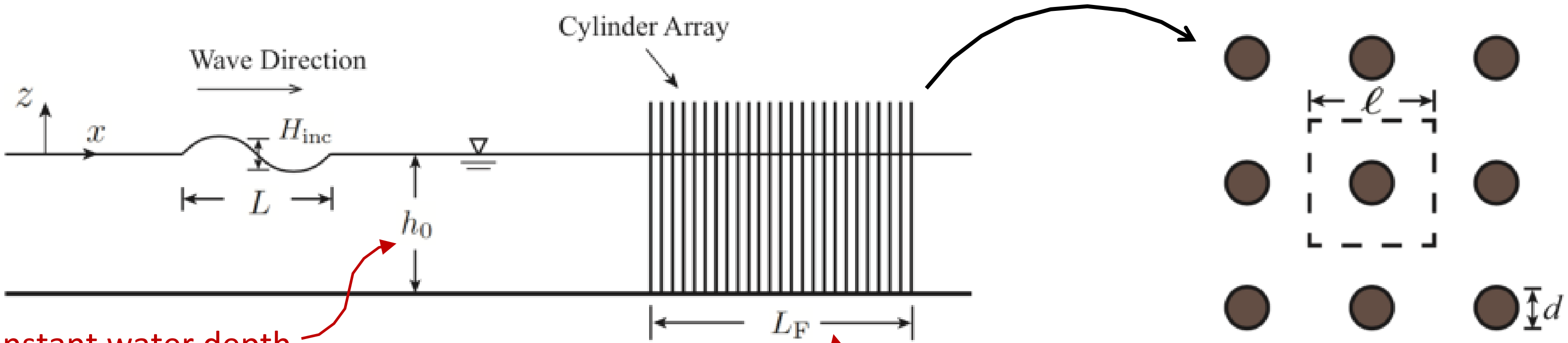
ν_e : eddy viscosity ← Spatial average

Incident waves:

simple-harmonic waves with weak nonlinearity



Periodic shallow-water waves



Constant water depth

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u_i : velocity components

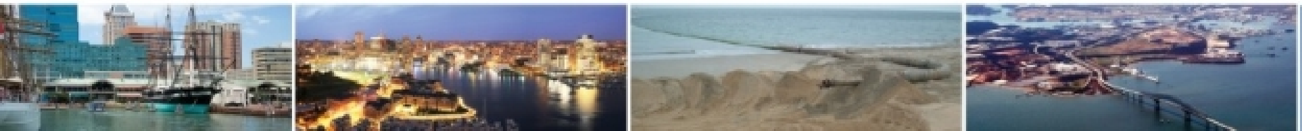
η : free surface elevation

ν_e : eddy viscosity ← Spatial average

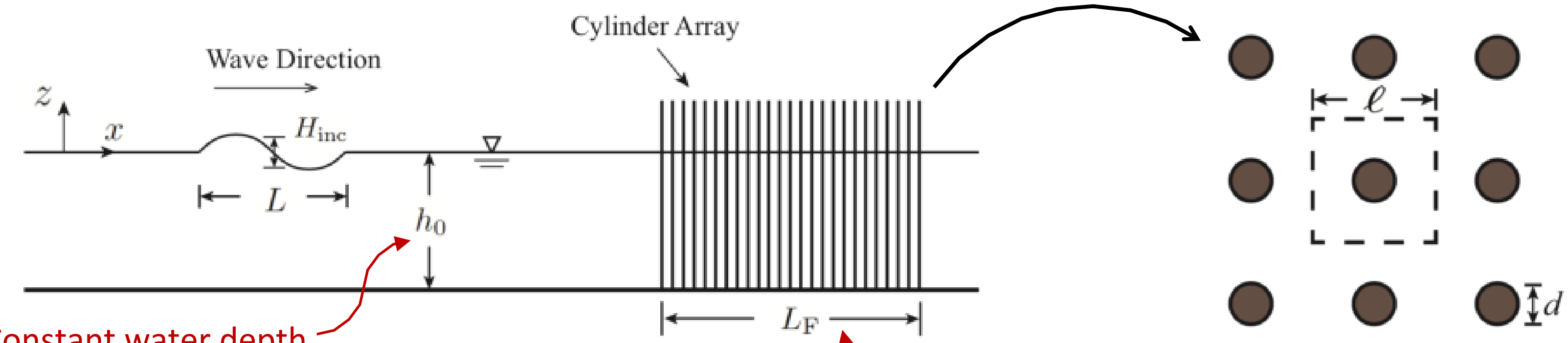
- Parameters:

$$\varepsilon = k_{\mathrm{inc}}\ell = \frac{\omega \ell}{\sqrt{g h_0}} \ll \mathcal{O}(1), \quad \alpha = \frac{H_{\mathrm{inc}}/2h_0}{\varepsilon} = \mathcal{O}(1) \Rightarrow \boxed{\mathcal{O}\left(\frac{H_{\mathrm{inc}}}{2h_0}\right) = \mathcal{O}(\varepsilon)}$$

Weakly nonlinear waves



Periodic shallow-water waves



Constant water depth

Emergent, rigid and vertical cylinders

Regular arrangements

- Governing equations (shallow water):

$$\frac{\partial \eta}{\partial t} + \frac{\partial}{\partial x_i}\left[u_i\left(h_0 + \eta\right)\right] = 0, \quad i = 1,\ 2$$

$$\frac{\partial u_i}{\partial t} + \boxed{u_j \frac{\partial u_i}{\partial x_j}} = -g\frac{\partial \eta}{\partial x_i} + \nu_e \frac{\partial^2 u_i}{\partial x_j \partial x_j}, \quad i \,\&\, j = 1,\ 2$$

u_i : velocity components

η : free surface elevation

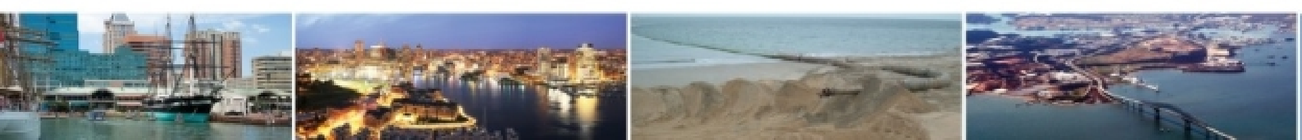
ν_e : eddy viscosity ← Spatial average

  - Eddy viscosity (Mei et al. 2011):

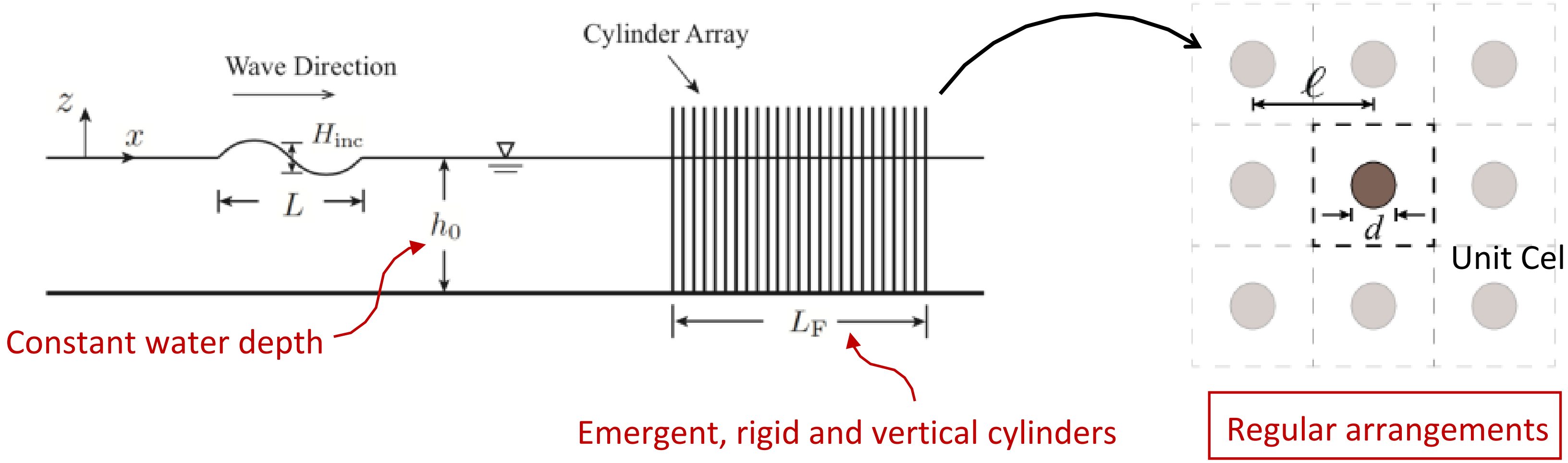
$$\nu_e = 1.86(1-n)^{2.06} U_0 \ell, \quad U_0 = \sqrt{g h_0} A_{\text{inc}}/h_0$$

porosity

Shallow-water wave characteristic velocity



Homogenization (multi-scale perturbation theory)

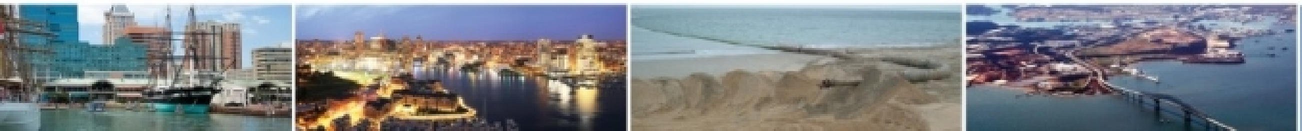


Macro-scale Problem	Micro-scale Problem
< scale: wavelength>	< scale: tree spacing >
$X_i^* = \varepsilon x_i^*$	$x_i^* = x_i/\ell$

$$\varepsilon = k_{\mathrm{inc}}\ell = \frac{\omega \ell}{\sqrt{g h_0}} \ll O(1)$$

dimensionless

$$(u_i^*, \eta^*) = (u_i^{*^{(0)}}, \eta^{*^{(0)}}) + \varepsilon(u_i^{*^{(1)}}, \eta^{*^{(1)}}) + \cdots \implies u_i^{*^{(n)}}, \eta^{*^{(n)}} : \text{ functions of } (x_i^*, X_i^*, t^*)$$

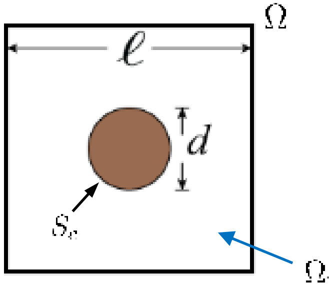


Leading-order problem

- Summation of different harmonics

$$u_i = \frac{1}{2}\sum_{\mathrm{m}=-\infty}^{\infty} \tilde{u}_{i,\mathrm{m}} e^{-i\mathrm{m}t} \qquad \eta = \frac{1}{2}\sum_{\mathrm{m}=-\infty}^{\infty} \tilde{\eta}_{\mathrm{m}} e^{-i\mathrm{m}t} \longrightarrow (\tilde{u}_{i,-\mathrm{m}},\ \tilde{\eta}_{-\mathrm{m}}) = \text{complex conjugate of } (\tilde{u}_{i,\mathrm{m}},\ \tilde{\eta}_{\mathrm{m}})$$

- Micro-scale (cell) problem - NONLINEAR



$$n = 1 - \frac{\pi d^2}{4\ell^2}$$

$$\frac{\partial \tilde{u}^{(0)}_{i,\mathrm{m}}}{\partial x_i} = 0, \quad \vec{x} \in \Omega_f$$

Macro-scale pressure gradient

$$(-i\mathrm{m})\,\tilde{u}^{(0)}_{i,\mathrm{m}} + \frac{\alpha}{2}\boxed{\sum_{\mathrm{m}_1=-\infty}^{\infty}\left(\tilde{u}^{(0)}_{j,\mathrm{m}_1}\frac{\partial \tilde{u}^{(0)}_{i,\mathrm{m}-\mathrm{m}_1}}{\partial x_j}\right)} = -\frac{\partial \tilde{\eta}^{(0)}_{\mathrm{m}}}{\partial X_i} - \frac{\partial \tilde{\eta}^{(1)}_{\mathrm{m}}}{\partial x_i} + \sigma\frac{\partial^2 \tilde{u}^{(0)}_{i,\mathrm{m}}}{\partial x_j \partial x_j}, \quad \vec{x} \in \Omega_f$$

Harmonic generation

Dimensionless eddy viscosity:

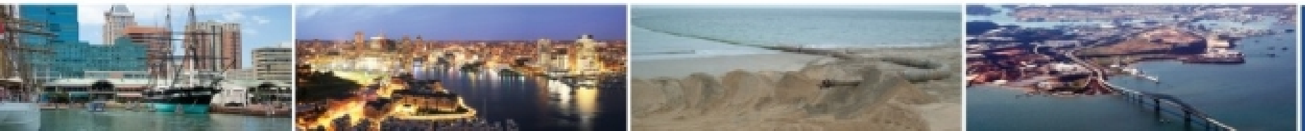
$$\sigma = \frac{\nu_e}{\omega \ell^2} \equiv 1.86(1-n)^{2.06}\frac{1}{k_{\mathrm{inc}}\ell}\left(\frac{A_{\mathrm{inc}}}{h_0}\right), \quad k_{\mathrm{inc}} = \frac{\omega}{\sqrt{g h_0}}$$

➢ Boundary conditions:

$$\tilde{u}^{(0)}_{i,\mathrm{m}} = 0, \quad \vec{x} \in S_c$$

$$\langle \tilde{\eta}^{(1)}_{\mathrm{m}} \rangle = \frac{1}{\Omega}\iint_{\Omega_f} \tilde{\eta}^{(1)}_{\mathrm{m}}\, dx_1 dx_2 = 0$$

Nonlinear B.V.P – Unknowns: $\tilde{u}^{(0)}_{i,\mathrm{m}}$ and $\tilde{\eta}^{(1)}_{\mathrm{m}}$



Leading-order problem

- Micro-scale (cell) problem - NONLINEAR

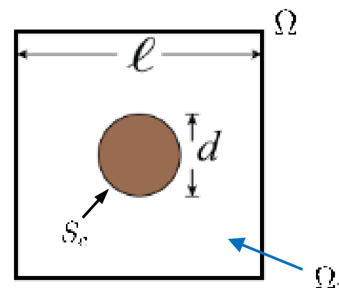
  - Modified pressure correction method – iteration

Dimensionless eddy viscosity

$$\frac{\partial \tilde{u}^{(0)}_{i,m}}{\partial t} - \mathrm{i}m\tilde{u}^{(0)}_{i,m} + \frac{\alpha}{2}\sum_{m_1=-\infty}^{\infty}\frac{\partial\left(\tilde{u}^{(0)}_{j,m_1}\tilde{u}^{(0)}_{i,m-m_1}\right)}{\partial x_j} - \frac{\partial \tilde{\eta}^{(0)}_m}{\partial X_i} - \frac{\partial \tilde{\eta}^{(1)}_m}{\partial x_i} + \sigma\frac{\partial^2 \tilde{u}^{(0)}_{i,m}}{\partial x_j \partial x_j}$$

Pseudo-time derivative

Macro-scale pressure gradient (GIVEN)

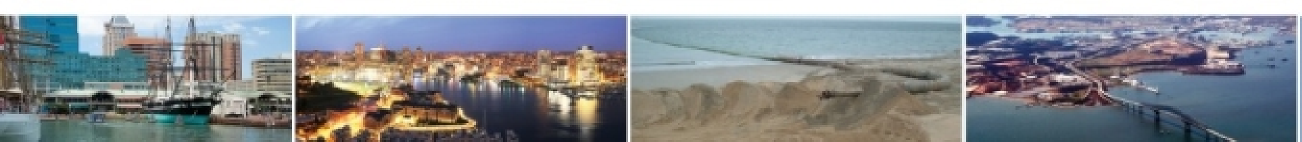


$$n = 1 - \frac{\pi d^2}{4\ell^2}$$

  - Finite difference with staggered discretization:

$$\frac{\left(\tilde{u}^{(0)}_{i,m}\right)^{n_t+1} - \left(\tilde{u}^{(0)}_{i,m}\right)^{n_t}}{\Delta t} = \mathrm{i}m\left(\tilde{u}^{(0)}_{i,m}\right)^{n_t} - \frac{\alpha}{2}\sum_{m_1=-\infty}^{\infty}\left(\frac{\partial \tilde{u}^{(0)}_{j,m_1}\tilde{u}^{(0)}_{i,m-m_1}}{\partial x_j}\right)^{n_t} - \frac{\partial \tilde{\eta}^{(0)}_m}{\partial X_i} - \left(\frac{\partial \tilde{\eta}^{(1)}_m}{\partial x_i}\right)^{n_t} + \sigma\left(\frac{\partial^2 \tilde{u}^{(0)}_{i,m}}{\partial x_j \partial x_j}\right)^{n_t}$$

if convergent → $\left(\tilde{u}^{(0)}_{i,m}\right)^{n_t+1} \approx \left(\tilde{u}^{(0)}_{i,m}\right)^{n_t}$ → $(n_t)^{\mathrm{th}}$ iteration



Leading-order problem

- Macro-scale (wavelength-scale) problem
  - Forest region for each harmonic – LINEAR

cell-averaged quantity

$$n\left(-\mathrm{i}\mathfrak{m}\tilde{\eta}_{\mathfrak{m}}^{(0)}\right)+\frac{\partial\langle\tilde{u}_{i,\mathfrak{m}}^{(0)}\rangle}{\partial X_i}=0$$

$$-\mathrm{i}\mathfrak{m}\langle\tilde{u}_{i,\mathfrak{m}}^{(0)}\rangle+\alpha\tilde{M}_{\mathfrak{m}}=-n\frac{\partial\tilde{\eta}_{\mathfrak{m}}^{(0)}}{\partial X_i}-\tilde{N}_{\mathfrak{m}}+\sigma\tilde{Q}_{\mathfrak{m}}$$

$$\Rightarrow\quad \frac{\partial^2\tilde{\eta}_{\mathfrak{m}}^{(0)}}{\partial X_i^2}+\left(\mathfrak{m}^2\right)\tilde{\eta}_{\mathfrak{m}}^{(0)}=-\left(\frac{\alpha}{n}\right)\frac{\partial\tilde{M}_{\mathfrak{m}}}{\partial X_i}-\left(\frac{1}{n}\right)\frac{\partial\tilde{N}_{\mathfrak{m}}}{\partial X_i}+\left(\frac{\sigma}{n}\right)\frac{\partial\tilde{Q}_{\mathfrak{m}}}{\partial X_i}$$

  - Complex coefficients: ← Cell problem solutions

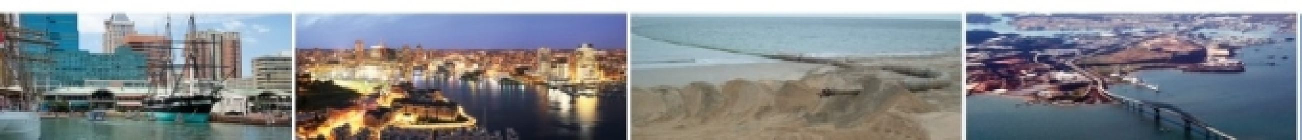
$$\tilde{M}_{\mathfrak{m}}=\frac{1}{2\Omega}\iint_{\Omega_f}\left[\sum_{\mathfrak{m}_1=-\infty}^{\infty}\tilde{u}_{j,\mathfrak{m}_1}^{(0)}\frac{\partial\tilde{u}_{i,\mathfrak{m}-\mathfrak{m}_1}^{(0)}}{\partial x_j}\right]d\Omega,\quad \tilde{N}_{\mathfrak{m}}=\frac{1}{\Omega}\iint_{\Omega_f}\frac{\partial\tilde{\eta}_{\mathfrak{m}}^{(1)}}{\partial x_i}d\Omega,\quad \tilde{Q}_{\mathfrak{m}}=\frac{1}{\Omega}\iint_{\Omega_f}\frac{\partial^2\tilde{u}_{i,\mathfrak{m}}^{(0)}}{\partial x_j\partial x_j}d\Omega$$

  - Open water region for each harmonic – LINEAR

$$-\mathrm{i}\mathfrak{m}\tilde{\eta}_{\mathfrak{m}}^{(0)}+\frac{\partial\langle\tilde{u}_{i,\mathfrak{m}}^{(0)}\rangle}{\partial X_i}=0$$

$$-\mathrm{i}\mathfrak{m}\langle\tilde{u}_{i,\mathfrak{m}}^{(0)}\rangle=-\frac{\partial\tilde{\eta}_{\mathfrak{m}}^{(0)}}{\partial X_i}$$

$$\Rightarrow\quad \frac{\partial^2\tilde{\eta}_{\mathfrak{m}}^{(0)}}{\partial X_i^2}+\left(\mathfrak{m}^2\right)\tilde{\eta}_{\mathfrak{m}}^{(0)}=0$$



Leading-order problem

- Micro-scale problem:

$$\frac{\partial \tilde{u}^{(0)}_{i,m}}{\partial x_i} = 0, \quad \vec{x} \in \Omega_f$$

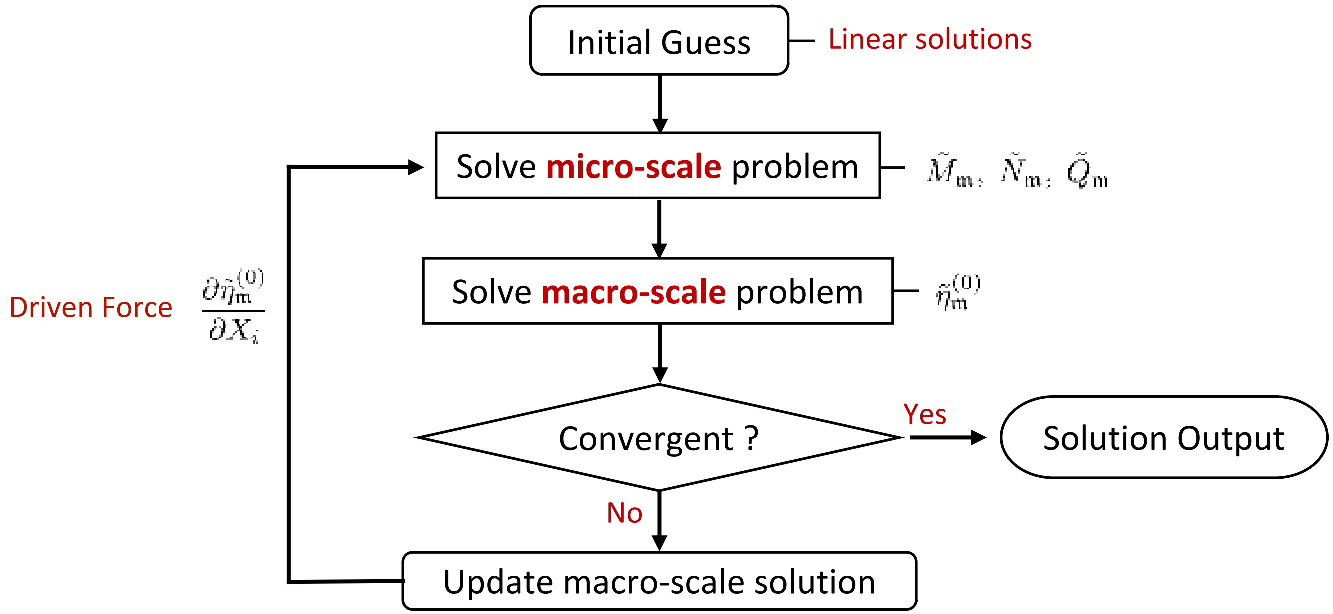
Macro-scale pressure gradient

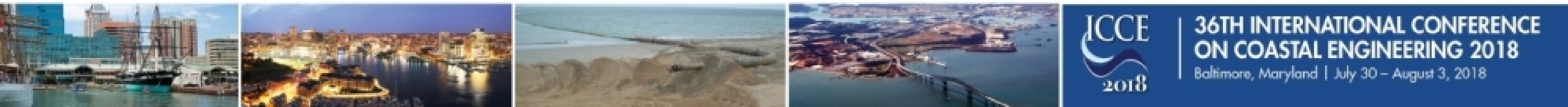
$$(-im)\,\tilde{u}^{(0)}_{i,m} + \frac{\alpha}{2} \sum_{m_1=-\infty}^{\infty} \left(\tilde{u}^{(0)}_{j,m_1} \frac{\partial \tilde{u}^{(0)}_{i,m-m_1}}{\partial x_j} \right) = -\frac{\partial \tilde{\eta}^{(0)}_m}{\partial X_i} - \frac{\partial \tilde{\eta}^{(1)}_m}{\partial x_i} + \sigma \frac{\partial^2 \tilde{u}^{(0)}_{i,m}}{\partial x_j \partial x_j}, \quad \vec{x} \in \Omega_f$$

- Macro-scale problem:

$$\frac{\partial^2 \tilde{\eta}^{(0)}_m}{\partial X_i^2} + (m^2)\,\tilde{\eta}^{(0)}_m = -\left(\frac{\alpha}{n}\right) \boxed{\frac{\partial \tilde{M}_m}{\partial X_i}} - \left(\frac{1}{n}\right) \boxed{\frac{\partial \tilde{N}_m}{\partial X_i}} + \left(\frac{\sigma}{n}\right) \boxed{\frac{\partial \tilde{Q}_m}{\partial X_i}}$$

Cell problem solutions





Long waves through a forest belt

- Macro-scale (wavelength-scale) problem

  - Forest region: $\frac{\partial^2 \tilde{\eta}_m^{(0)}}{\partial X^2} + (m^2)\,\tilde{\eta}_m^{(0)} = -\left(\frac{\alpha}{n}\right)\frac{\partial \tilde{M}_m}{\partial X} - \left(\frac{1}{n}\right)\frac{\partial \tilde{N}_m}{\partial X} + \left(\frac{\sigma}{n}\right)\frac{\partial \tilde{Q}_m}{\partial X}, \quad \text{if} \quad 0 < X < L_F$

  - Open water: $\frac{\partial^2 \tilde{\eta}_m^{(0)}}{\partial X^2} + (m^2)\,\tilde{\eta}_m^{(0)} = 0, \quad \text{if} \quad X < 0 \quad \text{or} \quad X > L_F$

$$\tilde{\eta}_{I,m}^{(0)} = \mathcal{I}_m e^{imX} + \mathcal{R}_m e^{-imX}, \quad \tilde{u}_{I,m}^{(0)} = \mathcal{I}_m e^{imX} - \mathcal{R}_m e^{-imX} \quad \text{if} \quad X < 0$$

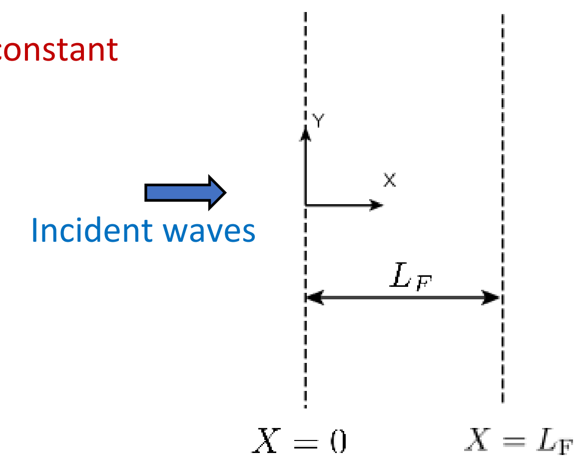
$$\tilde{\eta}_{T,m}^{(0)} = \mathcal{T}_m e^{imX}, \quad \tilde{u}_{T,m}^{(0)} = \mathcal{T}_m e^{imX} \quad \text{if} \quad X > L_F$$

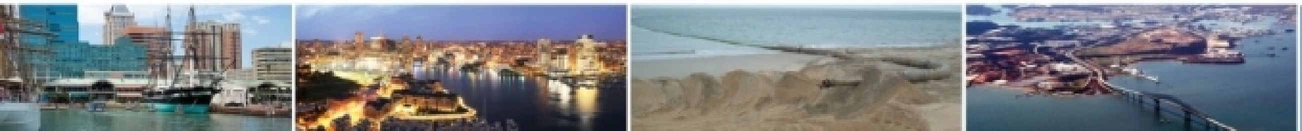
$R_0 = 0$ → Mean water level of incidence region = constant

  - Matching conditions:

$$\eta_{I,m}^{(0)} = \eta_{F,m}^{(0)}; \quad \tilde{u}_{I,m}^{(0)} = \tilde{u}_{F,m}^{(0)} \quad \text{at} \quad X = 0$$

$$\eta_{F,m}^{(0)} = \eta_{T,m}^{(0)}, \quad \tilde{u}_{F,m}^{(0)} = \tilde{u}_{T,m}^{(0)} \quad \text{at} \quad X = L_F$$





Long waves through a forest belt

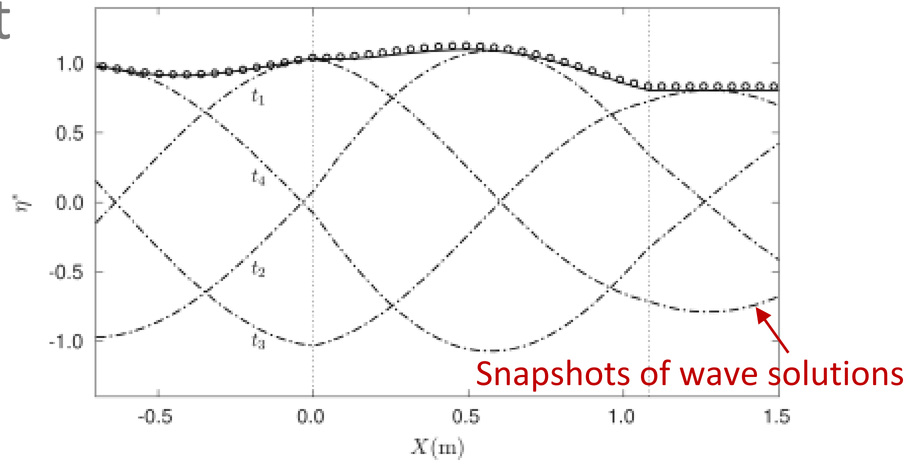
- Case 1: $T = 2.50\,\mathrm{s},\ A_{\mathrm{inc}} = 0.33\,\mathrm{cm}$

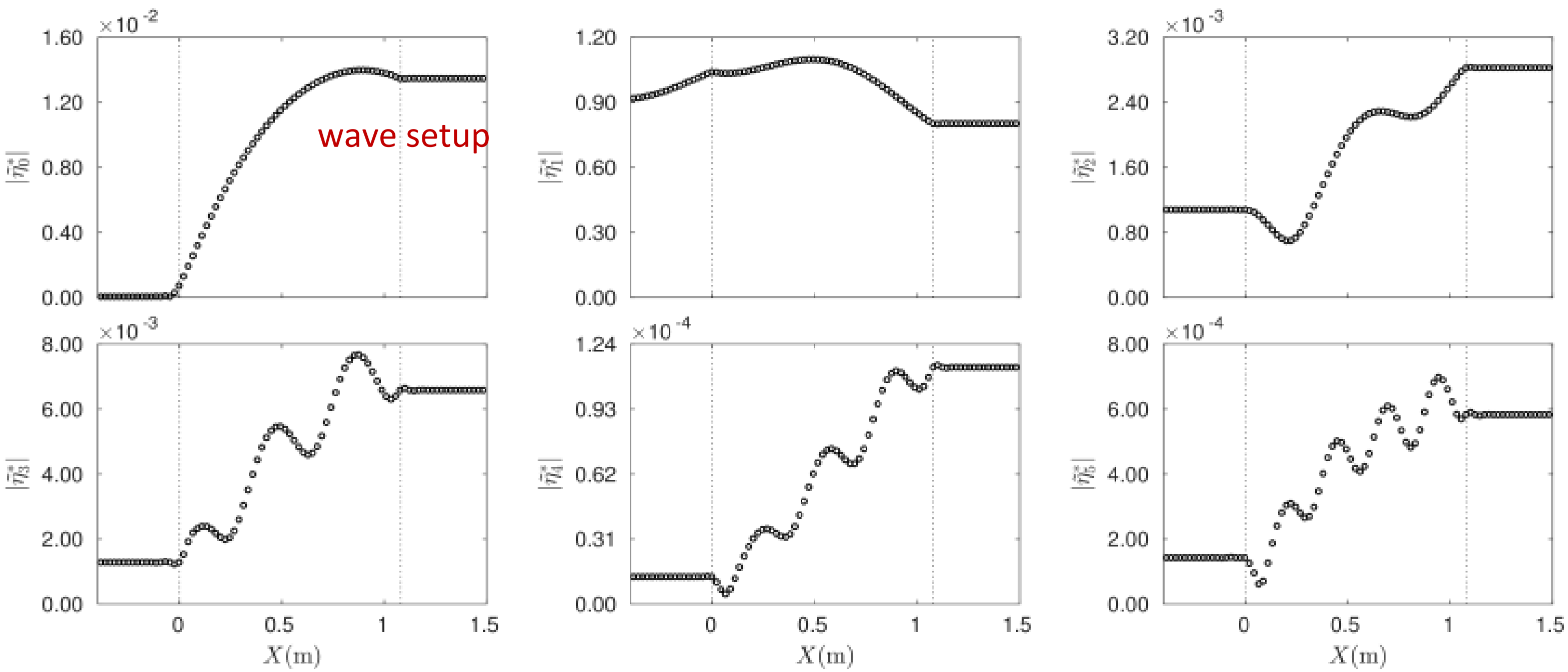
  $h_0 = 12\,\mathrm{cm},\ L = 2.692\,\mathrm{m}$

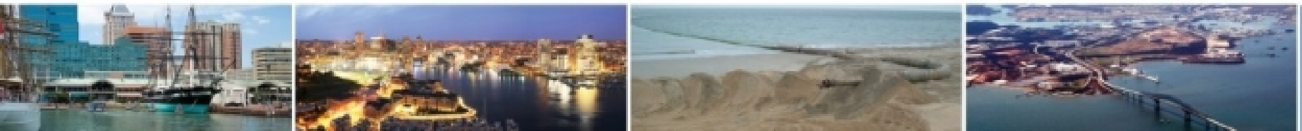
  → $\alpha = 0.3957,\ \sigma = 0.0048$

Circles — linear model solution

Solid line — upper wave envelope







Long waves through a forest belt

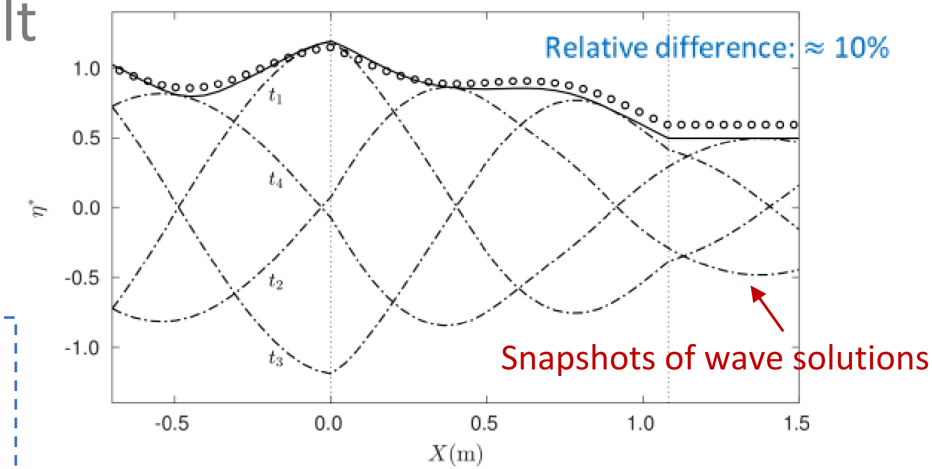
- Case 2: $T = 1.90\,\mathrm{s},\ A_{\mathrm{inc}} = 1.22\,\mathrm{cm}$

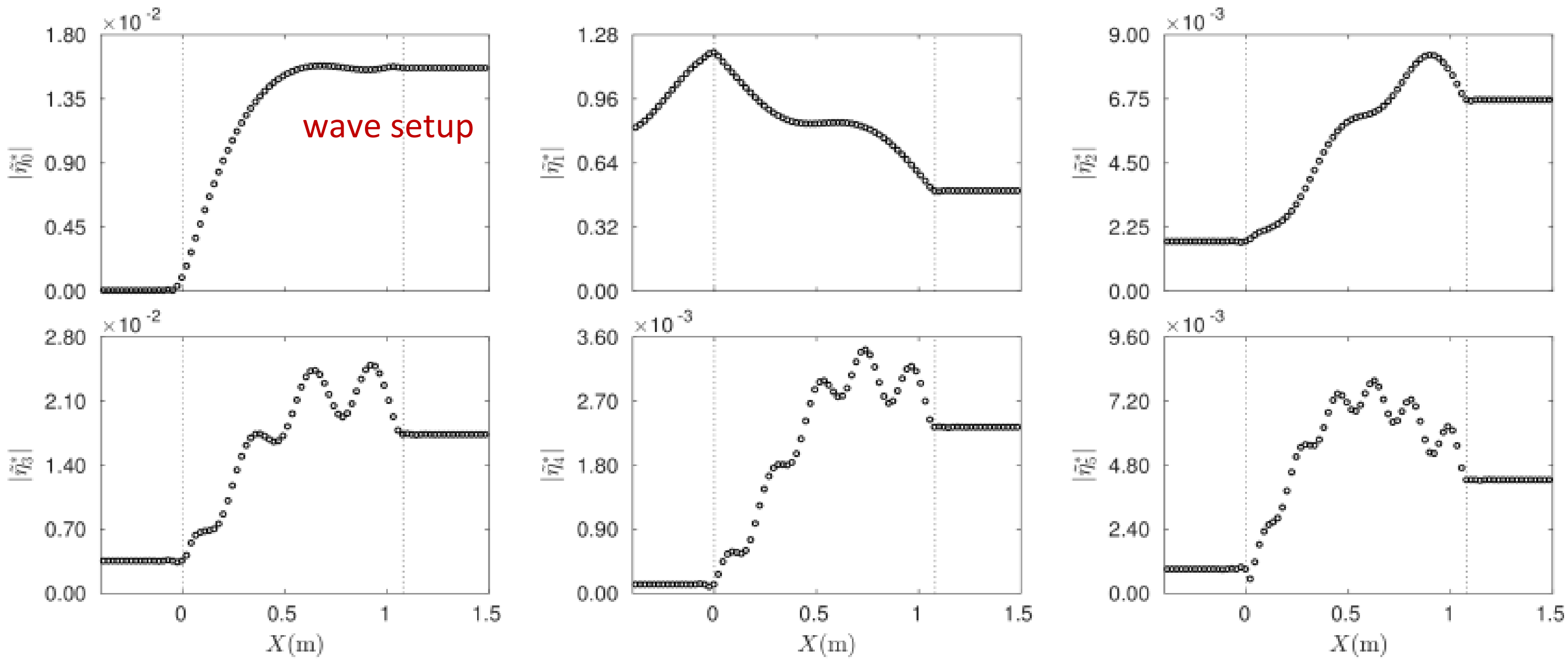
  $h_0 = 12\,\mathrm{cm},\ L = 2.038\,\mathrm{m}$

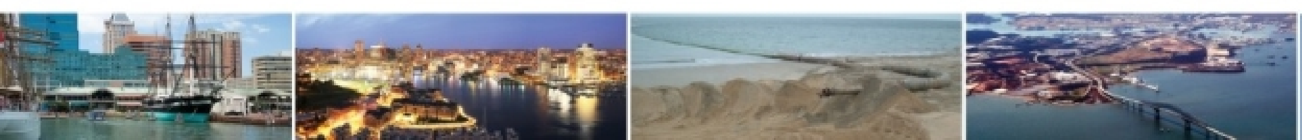
  → $\alpha = 1.1119,\ \sigma = 0.0136$

Circles — linear model solution

Solid line — upper wave envelope



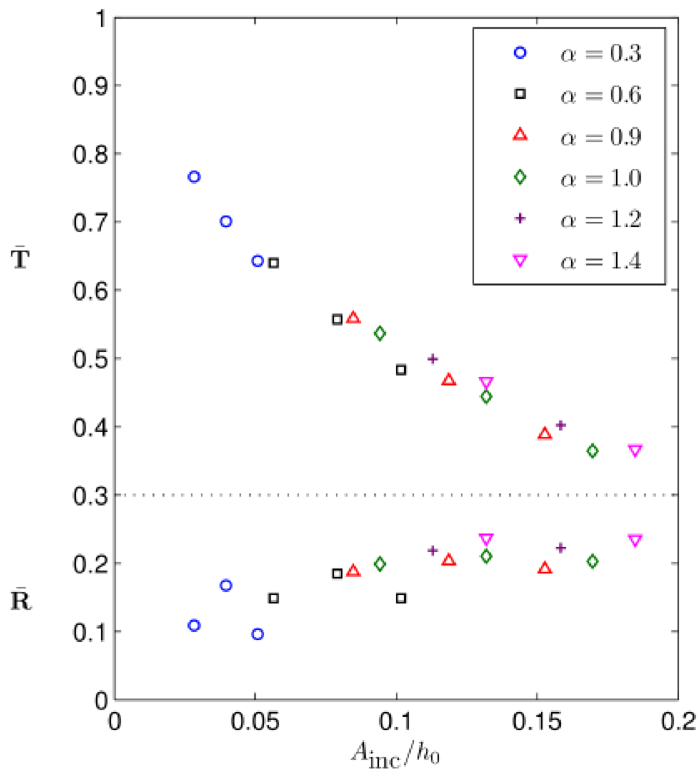


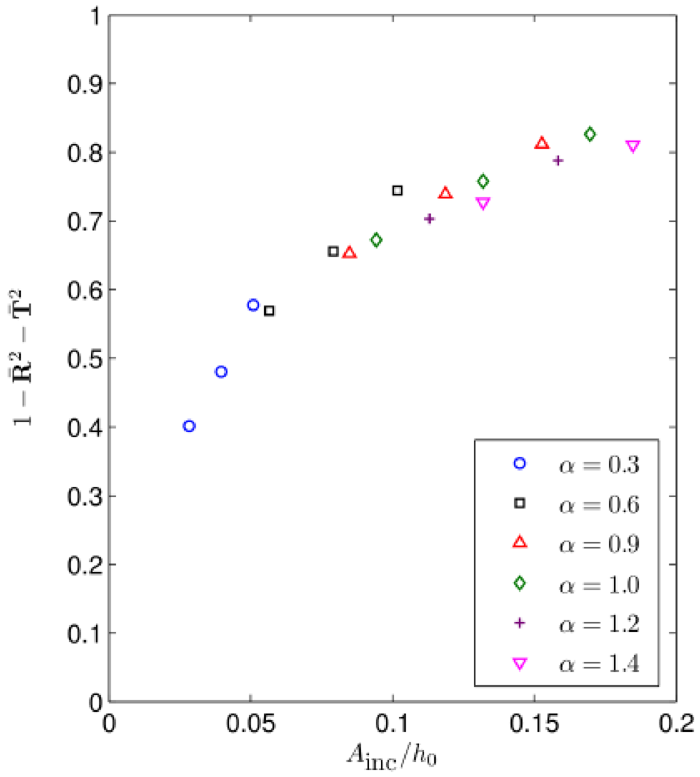


Long waves through a forest belt

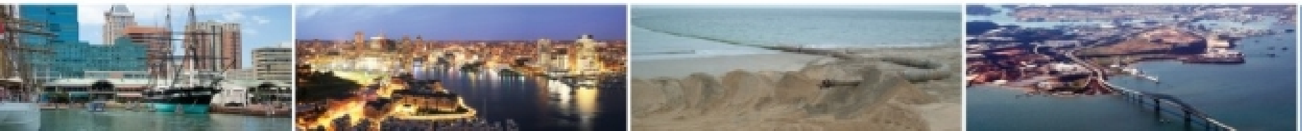
$$\alpha = \frac{A_{\text{inc}}/h_0}{k_{\text{inc}}\ell} = \frac{A_{\text{inc}}/h_0}{\varepsilon}$$

$$\sigma = 1.86(1-n)^{2.06}\frac{1}{k_{\text{inc}}\ell}\left(\frac{A_{\text{inc}}}{h_0}\right) \equiv 1.86(1-n)^{2.06}\cdot\alpha$$





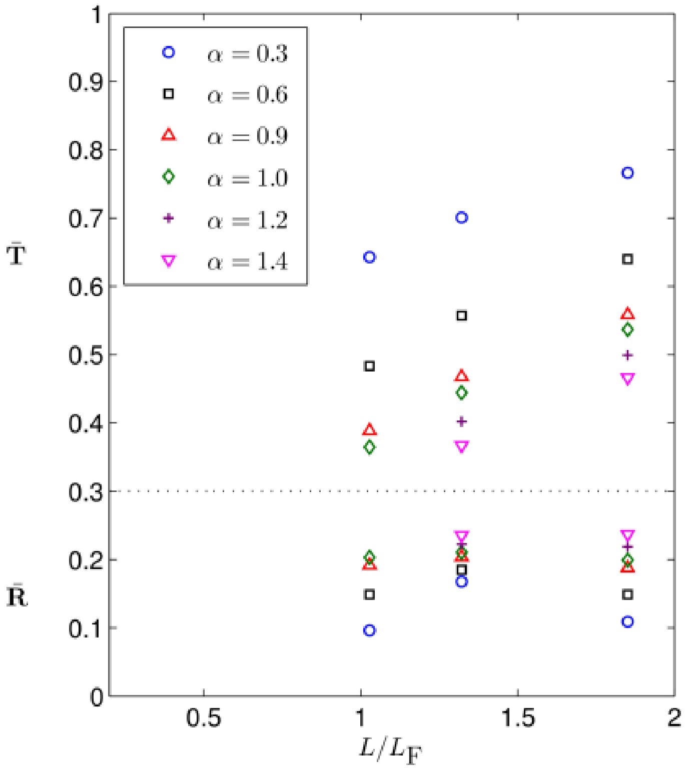
Reflection coefficient: $\bar{\mathbf{R}} = \left(\sum_{m=0}^{\infty}|\mathcal{R}_m|^2\right)^{1/2}$ Transmission coefficient: $\bar{\mathbf{T}} = \left(\sum_{m=0}^{\infty}|\mathcal{T}_m|^2\right)^{1/2}$

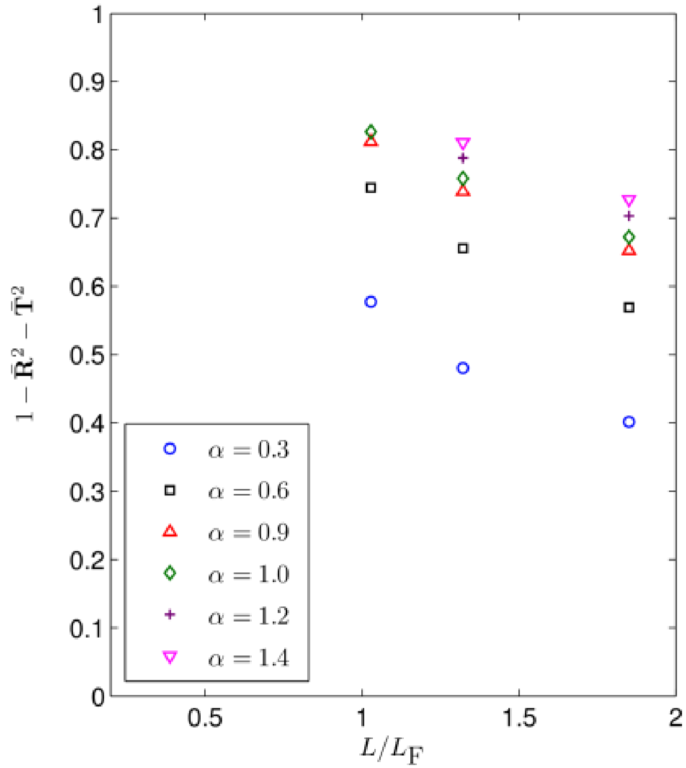


Long waves through a forest belt

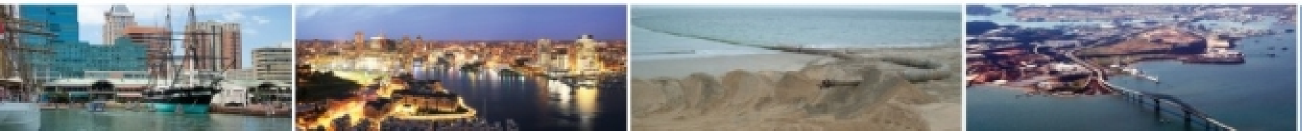
$$\alpha = \frac{A_{\rm inc}/h_0}{k_{\rm inc}\ell} = \frac{A_{\rm inc}/h_0}{\varepsilon}$$

$$\sigma = 1.86(1-n)^{2.06}\frac{1}{k_{\rm inc}\ell}\left(\frac{A_{\rm inc}}{h_0}\right) \equiv 1.86(1-n)^{2.06}\cdot\alpha$$





Reflection coefficient: $\bar{R} = \left(\sum_{m=0}^{\infty}|\mathcal{R}_m|^2\right)^{1/2}$ Transmission coefficient: $\bar{T} = \left(\sum_{m=0}^{\infty}|\mathcal{T}_m|^2\right)^{1/2}$



Summary

- Micro-scale problem:
  - ➢ The numerical model for solving micro-scale nonlinear problem is developed
  - ➢ Boundary-fitting discretization is needed for improvement
- Macro-scale problem:
  - ➢ Higher harmonics are generated and radiated into outside region
  - ➢ The first harmonic is dominant and higher harmonics have smaller amplitude
  - ➢ Lack of gauge data for shallow-water waves
    - ○ Model extension for taking vertical variation into account is needed
    - ○ The use of drag coefficient and eddy viscosity will be made
    - ○ More gauge data are available for model validation